First International Congress on Tools for Teaching Logic. Algebraic Logic and Probability Theory

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Abstract

Experience in the science is: connections between areas being relative far from each other are extremely fruitful. We deal with connections between algebraic logic and probability theory from the viewpoint of teaching at graduate or postgaraduate level. We sketch how can we use concepts from probability theory to demonstrate applications of logic and conversely how can we enlarge our tools to teach probability theory using concepts from logic. We remark that the terminologies „algebraic logic" and „probability theory" below can be changed by „logic" and „measure theory" respectively, but the …rst two seem to be more adequate for our purposes.

1. Boolean Set Algebras

Our starting point is the foundations of axiomatic probability theory. As it is known, in probability theory, in the simplest case, a Boolean set algebra B is given (algebra of events) and a probability measure is de…ned on B. The Boolean set algebra B represents in the theory the natural language and a simple logic e.g. propositional logic. So the background of using Boolean set algebras in probability theory is algebraic logic. This is why we can state that axiomatic probability theory is based on algebraic logic, really.

In general case, the domain of the probability measure is a Boolean ¾ set algebra instead of ordinary Boolean set algebra. From the viewpoint of logic this means that in…nitary propositional language and propositional logic is used instead of ordinary propositional language and logic, that is the language has in…nite conjunction, disjunction and there are formulas of in…nite length.

2. Algebras of …rst order elementary classes

The connection between …rst order logic and probability theory is less traditional one than the connection between propositional logic and probability theory. First order logic appear mainly in the theory of stochastic processes inside probability theory. In this theory appear …rst order properties, e.g.

monotonicity of a function $X(t)$, formally

$$8t8s(t < s \ ! \ X(t) \quad X(s)),$$

boundedness of a function $X(t)$; formally

$$9s8t(X(t) \quad s),$$

a function $X(t)$ takes n di¤erent values at most, formally

$$9s_1 : : : 9s_n 8t(X(t) = s_1 \_ : : : \_ X(t) = s_n);$$

where the function $X(t)$ is a realization of the process.

To assign probabilities to these properties or to the sets representing these properties require nontraditional methods because these sets proved to be nontraditional from the viewpoint of the traditional probability or measure theory. Special trickies are needed to reduce this extension problem to traditional results. To realize this extension problem we must analyse the logical background of the problem: traditional probability theory and measure theory applies Boolean set algebras representing propositional logic. The …rst order properties above and the problem assigning probabilities to these formulas require the usage of algebras representing …rst order logic.

As a formal device we need a usual …rst order in…nitary language L with a set of constants C [R; a set of unary relations Q_r; r 2 R; the relation < and a unary function symbol X: With C and R we can associate the parameter set C^0 of the given stochastic process and the set R^0 of the real numbers, respectively.

Let us consider all the models with universe R^0 corresponding to L and let B^C denote the ¾ set algebra of the restricted elementary classes corresponding to the closed formulas of L: Let A^C denote the ¾ set subalgebra of B^C corresponding to the closed quanti…er-free formulas of L: It is easy to check that traditional measures (Lebesgue, Borel, e.t.c.) can be considered to be de…ned on the algebra A^C: So our formulation of the extension problem is: how to extend the probability measure from the algebra A^C to the algebra B^C: We can show that the extension exists under suitable conditions. We are going to detail these conditions.

3. Cylindric Set Algebras

Algebras corresponding to ordinary …rst order models play an important role in probability and measure theory, too. With a …rst order model M; as we know, a cylindric set algebra D^M can be associated. The interest of this kind of algebra from the viewpoint of probability theory, measure theory, topology, e.t.c., that this algebra is closed to projection (cylindri…cation). If the universe of a model M is the set A and the set of the individuum variables in L is fx_c : c 2 Cg then, as it is known, the universe of the algebra D^M is the C-sequences from A and the elements of D^M are certain subsets of these sequences. Simple example for a cylindric set algebra without diagonals is the set of …nite unions of the …nite dimensional intervalls of R^{0C} with respect to the usual set operations:

We can associate a stochastic process with a cylindric set algebra D^M; as well. Namely, supposing a suitable …rst order language S let the individuum variables of S de…ne the random variables. Since a cylindric set algebra is a special Boolean algebra, we can de…ne a probability measure on it. It is remarkable that there is a kind of measure (Q-measure) de…ned on cylindric set algebras such that its de…nition is based, among others, on set operation projection.

Further possible connections between algebraic logic and probability theory are: connections between interpretations of models and measurable functions or connections between standard and nonstandard analysis, e.t.c.. We have no space here to detail these connections.

We hope that the connections mentioned in this lecture yield a contribution to the tools for teaching applied logic, and in a wider sense, to the tools for teaching mathematics in a complex way.

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